Non-classical logics: theory and applications
Part 2
Paraconsistent Logics

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Stanford Encyclopedia of Philosophy: The contemporary logical orthodoxy has it that, from contradictory premises, anything can be inferred. To be more precise, let $\models$ be a relation of logical consequence, defined either semantically or proof–theoretically. Call $\models$ explosive if it validates $\{A, \neg A\} \models B$ for every A and B (ex contradictione quodlibet).

The major motivation behind paraconsistent logic is to challenge this orthodoxy. A logical consequence relation, $\models$, is said to be paraconsistent if it is not explosive. Thus, if $\models$ is paraconsistent, then even if we are in certain circumstances where the available information is inconsistent, the inference relation does not explode into triviality. Thus, paraconsistent logic accommodates inconsistency in a sensible manner that treats inconsistent information as informative.

In Belnap's first order paraconsistent logic (1977), four possible values associated with a formula α are true, false, contradictory and unknown:

• if there is evidence for α and no evidence against α, then α obtains the value true and

• if there is no evidence for α and evidence against α, then α obtains the value false.

• A value contradictory corresponds to a situation where there is simultaneously evidence for α and against α and, finally,

• α is labeled by value unknown if there is no evidence for α nor evidence against α.

More formally, the values are associated with ordered couples $T = \langle 1, 0 \rangle$, $F = \langle 0, 1 \rangle$, $K = \langle 1, 1 \rangle$ and $U = \langle 0, 0 \rangle$, respectively.

Perny, Tsoukias and Öztürk introduced a $[0,1]$–valued extension of Belnap's logic: the graded values are computed via

$$t(\Phi) = \min\{\alpha, 1-\beta\}, \quad (1)$$

$$k(\Phi) = \max\{\alpha+\beta-1, 0\}, \quad (2)$$

$$u(\Phi) = \max\{1-\alpha-\beta, 0\}, \quad (3)$$

$$f(\Phi) = \min\{1-\alpha, \beta\}, \quad (4)$$

where $\langle\alpha, \beta\rangle$, called evidence couple, is given; α and β is the degree of evidence of a statement Φ and against Φ, respectively. Moreover, the set of 2×2 evidence matrices of a form

$$\begin{bmatrix} f(\Phi) & k(\Phi) \\ u(\Phi) & t(\Phi) \end{bmatrix}$$

is denoted by $\mathcal{M}$. The values $f(\Phi), k(\Phi), u(\Phi)$ and $t(\Phi)$ are values on $[0,1]$ such that $f(\Phi)+k(\Phi)+u(\Phi)+t(\Phi) = 1$. Truth and falsehood are not each others complements.

The operations in (1) – (4) are expressible in the Lukasiewicz structure, which is an example of an injective MV–algebra.
Lukasiewicz–Pavelka style fuzzy sentential logic is a complete logic (i.e. *a*–tautologies and *a*–provable formulae coincide).
We proved that, starting from a set of injective MV-algebra L valued evidence couples $\langle\alpha,\beta\rangle$, the structure of the evidence matrices

$$\begin{bmatrix} \alpha^* \wedge \beta & \alpha \odot \beta \\ \alpha^* \odot \beta^* & \alpha \wedge \beta^* \end{bmatrix} \tag{5}$$

forms an injective MV–algebra, too. Here the operations $\odot, \wedge$ and * are the algebraic operations product, meet and complement, respectively, of the original injective MV–algebra L.

Our result that continuous valued paraconsistent logic can be seen as a special case of Lukasiewicz–Pavelka style fuzzy logic has a consequence that a rich logical semantics and syntax is available. For example, all Lukasiewicz tautologies as well as Intuitionistic tautologies can be expressed in the framework of this logic. This follows by the fact that we have two sorts of logical connectives conjunction, disjunction, implication and negation interpreted either by the monoidal operations $\bigodot, \bigoplus, \longrightarrow, ^*$ or by the lattice operations $\wedge, \vee, \Rightarrow, ^\star$, respectively (however, neither * nor * is a lattice complementation). Besides, there are many other logical connectives available.

How is this paraconsistent fuzzy logic related to GUHA-logic?

Concider, for example, the following fancied allergy matrix:

Child	Tomato	Apple	Orange	Cheese	Milk
Anna	1	1	0	1	1
Aina	1	1	1	0	0
Naima	1	1	1	1	1
Rauha	0	1	1	0	1
Kai	0	1	0	1	1
Kille	1	1	0	0	1
Lempi	0	1	1	1	1
Ville	1	0	0	0	0
Ulle	1	1	0	1	1
Dulle	1	0	1	0	0
Dof	1	0	1	0	1
Kinge	0	1	1	0	1
Laade	0	1	0	1	1
Koff	1	1	0	1	1
Olavi	0	1	1	1	1

Here ϕ could mean child is allergic to tomato and apple and ψ could mean child is allergic to milk.

Recall the four–fold table

	ψ	$\neg\psi$
ϕ	a	b
$\neg\phi$	c	d

A statement connecting two attributes ϕ and ψ by basic double implicational quantifier is supported by the data or is TRUE if

$$a \geq n \text{ and } \frac{a}{a+b+c} \geq p,$$

where $n \in \mathcal{N}$ and $p \in [0, 1]$ are parameters given by user.

Our observation is that a value $\alpha = \frac{a}{m}$ can be seen as the degree of evidence that ϕ and ψ occur simultaneously, a value $\beta = \frac{b+c}{m}$ can be seen as the degree of evidence that ϕ and ψ do not occur simultaneously and a value $\frac{d}{m}$ the degree that ϕ and ψ do not occur at all – a kind of indifferent situation. Then

$$\alpha^* \wedge \beta = \beta,\ \alpha \odot \beta = 0,\ \alpha^* \odot \beta^* = \frac{d}{m},\ \alpha \wedge \beta^* = \alpha.$$

Therefore $\langle \frac{a}{m}, \frac{b+c}{m} \rangle$ can be seen as an evidence couple for a statement Φ: 'ϕ and ψ occur simultaneously'. The correspondent evidence matrix is then

$$\begin{bmatrix} f(\Phi) & k(\Phi) \\ u(\Phi) & t(\Phi) \end{bmatrix} = \begin{bmatrix} \frac{b+c}{m} & 0 \\ \frac{d}{m} & \frac{a}{m} \end{bmatrix}.$$

In practical data mining it happens that indifferent cases rule over interesting cases, i.e. value d in a four–fold contingency table is much bigger that values a, b, c. However, even in such cases it is useful to look for statements Φ such that the truth value of Φ is, say at least $k(> 1)$ times bigger than the falsehood of Φ, i.e. $\alpha \geq k\beta$, which is equivalent to $a \geq k(b + c)$. On the other hand such a statement Φ is stamped by label supported by the data if

$$\frac{a}{a+b+c} \geq p \text{ iff } a \geq \frac{p}{1-p}(b + c).$$

This means $k = \frac{p}{1-p}$, $p \neq 1$, or equivalently $p = \frac{k}{k+1}$. We have

Theorem

Given a data, all statements Φ such that the truth value of Φ is at least $k(> 1)$ times bigger than the falsehood of Φ in the sense of para- consistent logic, can be found by using basic double implicational quantifier and setting $p = \frac{k}{k+1}$.

Consider the above data about children's allergies. Let ϕ stand for child is allergic to cheese and ψ stand for child is allergic to milk. Compute the evidence matrix for the statement Φ: 'ϕ and ψ occur simultaneously'.

Solution. From the data matrix we get the following contingency table

	Milk	¬Milk
Cheese	8	0
¬Cheese	4	3

Thus, the evidence couple is $\langle \frac{8}{15}, \frac{4+0}{15} \rangle$, and the correspondent evidence matrix is

$$\left[\begin{array}{cc} f(\Phi) & k(\Phi) \\ u(\Phi) & t(\Phi) \end{array} \right] = \left[\begin{array}{cc} \frac{4}{15} & 0 \\ \frac{3}{15} & \frac{8}{15} \end{array} \right]$$

The truth of cheese allergy and milk allergy occur simultaneously is two times bigger than the falsehood, the data supports Φ.

Lukasiewicz–Pavelka fuzzy logic as well as paraconsistent logic and GUHA logic are sound and theoretically well established, widely studied and acknowledged non–classical logics. Therefore all connections between these approaches clear the road for future investigation into better understanding of e.g. in data analysis, knowledge extraction, decision theory and modeling real world problems. Here we have shown some connections between these logics.

Our future aim is to develop further the connection between fuzzy logic, paraconsistent logic and GUHA data mining logic.